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Anna Kazeykina, Roman Novikov

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A. V. Kazeykina[1] and R. G. Novikov[2]

Abstract. In the present paper we begin studies on the large time asymptotic behavior for solutions of the Cauchy problem for the Novikov–Veselov equation (an analog of KdV in $2+1$ dimensions) at positive energy. In addition, we are focused on a family of reflectionless (transparent) potentials parameterized by a function of two variables. In particular, we show that there are no isolated soliton type waves in the large time asymptotics for these solutions in contrast with well-known large time asymptotics for solutions of the KdV equation with reflectionless initial data.

1 Introduction

We consider the scattering problem for the two-dimensional Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{fixed} > 0 \tag{1.1}$$

at a fixed positive energy, where

$$\begin{aligned} &v(x) = \overline{v(x)}, \quad v \in L^\infty(\mathbb{R}^2), \\ &|v(x)| < q(1+|x|)^{-2-\varepsilon}, \quad \varepsilon > 0,\ q > 0. \end{aligned} \tag{1.2}$$

It is known that for any $k \in \mathbb{R}^2$, such that $k^2 = E$, there exists a unique bounded solution $\psi^+(x,k)$ of equation (1.1) with the following asymptotics

$$\begin{aligned} \psi^+(x,k) = e^{ikx} - i\pi\sqrt{2\pi}\, e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + \\ + o\left(\frac{1}{\sqrt{|x|}}\right), \quad |x| \to +\infty. \end{aligned} \tag{1.3}$$

This solution describes scattering of incident plane wave e^{ikx} on the potential $v(x)$. The function $f = f(k,l)$, $k \in \mathbb{R}^2$, $l \in \mathbb{R}^2$, $k^2 = l^2 = E$, arising in (1.3), is the scattering amplitude for $v(x)$ in the framework of equation (1.1).

[1]Centre des Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, 91128, France email: kazeykina@cmap.polytechnique.fr

[2]Centre des Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, 91128, France email: novikov@cmap.polytechnique.fr

In the present paper we are focused on transparent (or invisible) potentials v for equation (1.1). We say that v is transparent if its scattering amplitude f is identically zero at fixed energy E. We consider transparent potentials for equation (1.1) as analogs of reflectionless potentials for the one–dimensional Schrödinger equation at all positive energies; see, for example, [1], [2] as regards reflectionless potentials in dimension one.

In [3] it was shown that

1. There are no nonzero transparent potentials for equation (1.1), where
$$v(x) = \overline{v(x)}, \quad v \in L^{\infty}(\mathbb{R}^2), \quad |v(x)| < \alpha e^{\beta|x|}, \quad \alpha > 0, \beta > 0. \quad (1.4)$$

2. There is a family of nonzero transparent potentials for equation (1.1), where
$$v(x) = \overline{v(x)}, \quad v \in \mathcal{S}(\mathbb{R}^2), \quad (1.5)$$
and $\mathcal{S}$ denotes the Schwartz class.

In the present paper, in addition to the scattering problem for (1.1), we consider its isospectral deformation generated by the following $(2+1)$–dimensional analog of the KdV equation:

$$\begin{gathered}\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),\\ \partial_{\bar z} w = -3\partial_z v, \quad v = \bar v,\\ v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},\end{gathered} \quad (1.6)$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Equation (1.6) is contained implicitly in the paper of S.V. Manakov [4] as an equation possessing the following representation:

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E), \quad (1.7)$$

(Manakov $L-A-B$ triple), where $L = -\Delta + v(x,t)$ or, in other words, L (at fixed t) is the Schrödinger operator of (1.1), A and B are suitable differential operators of the third and the zero order respectively. Equation (1.6) was written in an explicit form by S.P. Novikov and A.P. Veselov in [5], [6], where higher analogs of (1.6) were also constructed.

Note that both Kadomtsev–Petviashvili equations can be obtained from (1.6) by considering an appropriate limit $E \to \pm\infty$ (V.E. Zakharov).

In terms of scattering data the nonlinear equation (1.6), where $E = E_{fixed} > 0$,

$$\begin{aligned} &v \text{ is sufficiently regular and has sufficient decay as } |x| \to \infty, \\ &w \text{ is decaying as } |x| \to \infty, \end{aligned} \tag{1.8}$$

takes the form (2.1)–(2.2) (see Section 2) and, in particular,

$$\frac{\partial f(k,l,t)}{\partial t} = 2i\left[k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2\right] f(k,l,t), \tag{1.9}$$

$k = (k_1, k_2) \in \mathbb{R}^2$, $l = (l_1, l_2) \in \mathbb{R}^2$, $k^2 = l^2 = E$, where $f(\cdot,\cdot,t)$ is the scattering amplitude for $v(\cdot,t)$. Equation (1.9) implies that the nonlinear evolution equation (1.6) under assumptions (1.8) preserves the transparency (or invisibility) property of $v(x,0)$ in the framework of the scattering problem for (1.1).

In the present paper we begin studies on the large time asymptotic behavior for solutions of the Cauchy problem for (1.6) under assumptions (1.8). We give a large time estimate for the family of solutions of (1.6) with $E = E_{fixed} > 0$ given by (2.11)–(2.13) and parameterized by a function of two variables. All potentials of this family are transparent at fixed t and E. In addition, this family contains all solutions of (1.6) with $E = E_{fixed} > 0$ such that:

- $v(\cdot,0) \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}$ denotes the Schwartz class, (1.10)
- $v(\cdot,0)$ is transparent for (1.1), (1.11)
- $v(\cdot,0)$ satisfies (1.2), where (1.12)

$$q < Q(E,\varepsilon) \quad \text{"small norm" condition}, \tag{1.13}$$

and Q is a special real function with the properties $Q(E,\varepsilon) > 0$ as $E > 0$, $\varepsilon > 0$, $Q(E,\varepsilon) \to +\infty$ for fixed $\varepsilon > 0$ as $E \to +\infty$,

- $v, w \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$, $\partial_x^j v(x,t) = O\left(|x|^{-3}\right)$, $w(x,t) = o(1)$, $|x| \to \infty$,

for $t \in \mathbb{R}$, $j \in (\mathbb{N} \bigcup 0)^2$.

The aforementioned family of solutions of (1.6) was considered for the first time in [3]. In the present work we prove the following estimate

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v)\ln(3+|t|)}{1+|t|}, \quad x \in \mathbb{R}^2, t \in \mathbb{R} \tag{1.14}$$

for each v of this family.

Estimate (1.14) implies that there are no isolated soliton type waves in the large time asymptotics for $v(x,t)$, in contrast with large time asymptotics for solutions of the KdV equation with reflectionless initial data.

Apparently, it is not difficult to obtain the estimate (1.14), where the right–hand side is replaced by $\frac{\mathrm{const}(v)}{1+|t|}$ and even to give precise expression for the leading term of the asymptotics of $v(x,t)$ as $t \to \infty$. However, already (1.14) in its present form implies the aforementioned absence of isolated soliton–type waves in the large–time asymptotics for $v(x,t)$.

Studies on the large time asymptotics for solutions of the Cauchy problem for the Kadomtsev–Petviashvili equations were fulfilled in [7], [8], [9].

Estimate (1.14) is proved in sections 3–4, using the stationary phase method, techniques developed in [3] and [9] and an analysis of some cubic algebraic equation depending on a complex parameter.

2 Transparent Potentials and Inverse Scattering Transform

In order to study the large time behavior of the family of transparent potentials described in the previous section, we will use the inverse scattering transform for the two-dimensional Schrödinger equation (1.1) described in [3].

First, we give the complete definition of scattering data for (1.1). Let $k \in \mathbb{C}^2$, $k^2 = E$, $\mathrm{Im}k \neq 0$ and $v(x)$ satisfy conditions (1.2) and (1.13). Then there exists a unique solution of (1.1) such that

$$\psi(x,k) = e^{ikx}(1 + o(1)), \quad |x| \to \infty.$$

It can be shown ([12]) that for $E \in \mathbb{R}$, $\mathrm{Im}k \neq 0$ the function $\psi(x,k)$ can be expanded as

$$\psi(k,x) = e^{ikx} - \pi \mathrm{sgn}(\mathrm{Im}k_2, \bar{k}_1) e^{ikx} \left(\frac{a(k)}{-k_2 x_1 + k_1 x_2} + \frac{e^{-2i\mathrm{Re}kx} b(k)}{-\bar{k}_2 x_1 + \bar{k}_1 x_2} + o\left(\frac{1}{|x|}\right)\right).$$

For $E > 0$ the function $b(k)$ is considered to be scattering data for (1.1) in addition to the scattering amplitude $f(k,l)$ arising in (1.3). It was shown in [10, 11, 12] that at fixed positive energy $f(k,l)$ and $b(k)$ uniquely determine the potential $v(x)$ satisfying (1.2), (1.13) (while $f(k,l)$ alone is insufficient for this purpose).

If potential $v(x,t)$ satisfies the Novikov–Veselov equation (1.6) under assumptions (1.8), then dynamics of the scattering data is described by the following formulas

$$\frac{\partial b(k,t)}{\partial t} = 2i\left[k_1^3 + \bar{k}_1^3 - 3k_1k_2^2 + 3\bar{k}_1\bar{k}_2^2\right] b(k,t), \quad k \in \mathbb{C}^2, \mathrm{Im}k \neq 0, k^2 = E, \tag{2.1}$$

$$\frac{\partial f(k,l,t)}{\partial t} = 2i\left[k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2\right] f(k,l,t), \quad k,l \in \mathbb{R}^2, k^2 = l^2 = E. \tag{2.2}$$

Equation (2.2) implies that the Novikov–Veselov equation preserves the property of transparency. It was also shown in [3] that this equation does not preserve, in general, the property of very fast decay of initial data. It can only be guaranteed that if $v(x,0) \in \mathcal{S}(\mathbb{R}^2)$ and satisfies (1.2), (1.13), then for every t we have $v(x,t) = O(|x|^{-3})$.

In the present paper we are concerned with transparent potentials, i.e. in the further considerations we assume that $f(k,l,t) \equiv 0$. We will also put

$$E = 1 \tag{2.3}$$

without loss of generality (the case of an arbitrary fixed positive energy may be reduced to (2.3) by scaling transformation). Along with the function $\psi(x,k)$ we will consider the function $\mu(x,k)$ related to $\psi(x,k)$ by the following expression

$$\psi(x,k) = e^{ikx}\mu(x,k). \tag{2.4}$$

It is also convenient in the two-dimensional scattering theory to introduce new notations

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad \lambda = k_1 + ik_2.$$

Then

$$k_1 = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right), \quad k_2 = \frac{i}{2}\left(\frac{1}{\lambda} - \lambda\right)$$

and we will consider that

$$\psi = \psi(z,\lambda,t), \quad \mu = \mu(z,\lambda,t), \quad b = b(\lambda,t).$$

In new notations the Schrödinger equation takes the form

$$L\psi = E\psi, \quad L = -4\partial_z\partial_{\bar{z}} + v(z,t), \quad z \in \mathbb{C},$$

equation (2.1) is written as

$$\frac{\partial b(\lambda,t)}{\partial t} = i\left(\lambda^3 + \frac{1}{\lambda^3} + \bar{\lambda}^3 + \frac{1}{\bar{\lambda}^3}\right) b(\lambda,t),$$

and (2.4) takes the form

$$\psi(z,\lambda,t) = e^{\frac{i}{2}(\lambda\bar z + z/\lambda)}\mu(z,\lambda,t).$$

We also note that in the present paper notation $f(z)$ does not imply that f is holomorphic on z, i.e. we omit the dependency on $\bar z$ in the notations.

Let a transparent potential $v(x,t)$ satisfy at $t=0$ conditions (1.2), (1.13). Then the function $\mu(z,\lambda,t)$ has the following properties (see [3]):

1. $\mu(z,\lambda,t)$ is continuous on $\lambda \in \mathbb{C}$;

2. $\forall\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, the function $\mu(z,\lambda,t)$ satisfies the equation

$$\frac{\partial \mu(z,\lambda,t)}{\partial \bar\lambda} = r(\lambda,z,t)\overline{\mu(z,\lambda,t)}, \tag{2.5}$$

   where

$$r(\lambda,z,t) = \exp(iS(\lambda,z,t))r(\lambda), \quad r(\lambda) = \frac{\pi \mathrm{sgn}(\lambda\bar\lambda - 1)}{\bar\lambda} b(\lambda,0), \tag{2.6}$$

$$S(\lambda,z,t) = \left\{-\frac{1}{2}\left(\bar\lambda z + \lambda\bar z + \frac{z}{\lambda} + \frac{\bar z}{\bar\lambda}\right)\right\} + \left\{t\left(\lambda^3 + \bar\lambda^3 + \frac{1}{\lambda^3} + \frac{1}{\bar\lambda^3}\right)\right\}; \tag{2.7}$$

3. $\mu(z,\lambda,t) \to 1$ as $\lambda \to 0, \infty$.

Properties 1–3 uniquely determine $\mu(z,\lambda,t)$ for all $\lambda \in \mathbb{C}$.

Under the same assumptions on the potential and if $v(x,0) \in \mathcal{S}(\mathbb{R}^2)$, the function $b(\lambda,t)$ has the following properties: for every $t \in \mathbb{R}$

$$b(\cdot,t) \in \mathcal{S}(\mathbb{C}); \tag{2.8}$$

$$b(1/\bar\lambda,t) = b(\lambda,t), \quad b(-1/\bar\lambda,t) = \overline{b(\lambda,t)}; \tag{2.9}$$

$$\partial_\lambda^m \partial_{\bar\lambda}^n b(\lambda,t)\big|_{|\lambda|=1} = 0 \quad \text{for all } m,n \geq 0. \tag{2.10}$$

The reconstruction of the transparent potential $v(z,t)$ from these scattering data is based on the following scheme.

1. Function $\mu(z,\lambda,t)$ is constructed as the solution of the following integral equation

$$\mu(z,\lambda,t) = 1 - \frac{1}{\pi}\iint_{\mathbb{C}} r(\zeta,z,t)\overline{\mu(z,\zeta,t)}\frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta - \lambda}. \tag{2.11}$$

which is uniquely solvable if the scattering data $b(\lambda, t)$ satisfy properties (2.8)–(2.10). Equation (2.11) is obtained from (2.5) by applying the Cauchy-Green formula

$$f(\lambda) = -\frac{1}{\pi}\iint\limits_{D} (\partial_{\bar\zeta} f(\zeta)) \frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta - \lambda} + \frac{1}{2\pi i}\oint\limits_{\partial D} f(\zeta)\frac{d\zeta}{\zeta - \lambda}.$$

2. Expanding $\mu(z, \lambda, t)$ as $\lambda \to \infty$,

$$\mu(z, \lambda, t) = 1 + \frac{\mu_{-1}(z, t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \tag{2.12}$$

we define $v(z, t)$ as

$$v(z, t) = 2i\partial_z \mu_{-1}(z, t). \tag{2.13}$$

3. It can be shown ([10]) that

$$L\psi = \psi$$

where

$$\psi(z, \lambda, t) = e^{\frac{i}{2}(\lambda\bar z + z/\lambda)}\mu(z, \lambda, t), \quad L = -4\partial_z\partial_{\bar z} + v(z, t),$$
$$\overline{v(z, t)} = v(z, t), \quad v(z, t) \text{ is transparent.}$$

3 Estimate for the linearized case

Consider

$$\begin{aligned} I(t, z) &= \iint\limits_{\mathbb{C}} f(\zeta)\exp(iS(\zeta, z, t))d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \\ J(t, z) &= -3\iint\limits_{\mathbb{C}} \frac{\bar\zeta}{\zeta} f(\zeta)\exp(iS(\zeta, z, t))d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \end{aligned} \tag{3.1}$$

where $f(\zeta) \in L^1(\mathbb{C})$, S is defined by (2.7). If $v(z, t) = I(t, z)$, $w(z, t) = J(t, z)$, where

$$(|\zeta|^3 + |\zeta|^{-3})f(\zeta) \in L^1(\mathbb{C})$$

as a function of ζ, and, in addition,

$$\overline{f(\zeta)} = f(-\zeta) \quad \text{and/or} \quad \overline{f(\zeta)} = -|\zeta|^{-4} f\left(-\frac{1}{\bar\zeta}\right),$$

then v, w satisfy the linearized Novikov–Veselov equation (1.6) with $E = 1$. In addition,

$$\hat{v}(p,t) \equiv 0 \quad \text{for} \quad |p| < 2, \quad t \in \mathbb{R},$$

where $\hat{v}(\cdot,t)$ is the Fourier transform of $v(\cdot,t)$, that is $v(\cdot,t)$ is transparent in the Born approximation at energy $E = 1$ for each $t \in \mathbb{R}$.

The goal of this section is to give, in particular, a uniform estimate of the large–time behavior of the integral $I(t,z)$ of (3.1) under the assumptions that

$$\begin{aligned} & f \in C^\infty(\mathbb{C}), \\ & \partial_\lambda^m \partial_{\bar\lambda}^n f(\lambda) = \begin{cases} O\left(|\lambda|^{-\infty}\right) & \text{as } |\lambda| \to \infty, \\ O\left(|\lambda|^{\infty}\right) & \text{as } |\lambda| \to 0, \end{cases} \\ & \partial_\lambda^m \partial_{\bar\lambda}^n f(\lambda)|_{|\lambda|=1} = 0 \end{aligned} \tag{3.2}$$

for all $m, n \geqslant 0$.

Applying the classical stationary phase method to (3.1), (3.2) (see, for example, [13]) yields

$$|I(t,z)| = O\left(\frac{1}{|t|}\right), t \to \infty, \tag{3.3}$$

uniformly on $z \in K$, where K is any compact set of the complex plane. This is not sufficient to guarantee the absence of soliton–type waves in the large time asymptotics of the potential $v(z,t) = I(t,z)$. So our further reasoning will be devoted to obtaining an estimate like (3.3) uniformly on $z \in \mathbb{C}$.

For this purpose we introduce parameter $u = \frac{z}{t}$ and write the integral I in the following form

$$I(t,u) = \iint\limits_{\mathbb{C}} f(\zeta) \exp(itS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{3.4}$$

where

$$S(u,\zeta) = -\frac{1}{2}\left(\bar\zeta u + \zeta \bar u + \frac{u}{\zeta} + \frac{\bar u}{\bar\zeta}\right) + \left(\zeta^3 + \bar\zeta^3 + \frac{1}{\zeta^3} + \frac{1}{\bar\zeta^3}\right). \tag{3.5}$$

We will start by studying the properties of the stationary points of the function $S(u,\zeta)$. These points satisfy the equation

$$S'_\zeta = -\frac{\bar u}{2} + \frac{u}{2\zeta^2} + 3\zeta^2 - \frac{3}{\zeta^4} = 0. \tag{3.6}$$

The degenerate stationary points obey additionally the equation

$$S''_{\zeta\zeta} = -\frac{u}{\zeta^3} + 6\zeta + \frac{12}{\zeta^5} = 0. \tag{3.7}$$

We denote $\xi = \zeta^2$ and

$$Q(u,\xi) = -\frac{\bar{u}}{2} + \frac{u}{2\xi} + 3\xi - \frac{3}{\xi^2}.$$

For each ξ, a root of the function $Q(u,\xi)$, there are two corresponding stationary points of $S(u,\zeta)$, $\zeta = \pm\sqrt{\xi}$.

The function $S'_\zeta(u,\zeta)$ can be represented in the following form

$$S'_\zeta(u,\zeta) = \frac{3}{\zeta^4}(\zeta^2 - \zeta_0^2(u))(\zeta^2 - \zeta_1^2(u))(\zeta^2 - \zeta_2^2(u)). \tag{3.8}$$

We will also use hereafter the following notations:

$$\mathcal{U} = \{u = 6(2e^{-i\varphi} + e^{2i\varphi}),\ \varphi \in [0, 2\pi)\}$$

and

$$\mathbb{U} = \{u = re^{i\varphi} \colon r \leq |6(2e^{-i\varphi} + e^{2i\varphi})|,\ \varphi \in [0, 2\pi)\},$$

the domain limited by the curve $\mathcal{U}$ (see also Figure 1).

Lemma 3.1.

1. *If $u = 18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, then*

$$\zeta_0(u) = \zeta_1(u) = \zeta_2(u) = e^{-\frac{\pi ik}{3}}$$

   and $S(u,\zeta)$ has two degenerate stationary points, corresponding to a third-order root of the function $Q(u,\xi)$, $\xi_1 = e^{-\frac{2\pi ik}{3}}$.

2. *If $u \in \mathcal{U}$ (i.e. $u = 6(2e^{-i\varphi} + e^{2i\varphi})$) and $u \neq 18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, then*

$$\zeta_0(u) = \zeta_1(u) = e^{i\varphi/2}, \quad \zeta_2(u) = e^{-i\varphi}.$$

   Thus $S(u,\zeta)$ has two degenerate stationary points, corresponding to a second-order root of the function $Q(u,\xi)$, $\xi_1 = e^{i\varphi}$, and two non–degenerate stationary points corresponding to a first-order root, $\xi_2 = e^{-2i\varphi}$.

3. *If $u \in \operatorname{int}\mathbb{U}$, then*

$$\zeta_i(u) = e^{i\varphi_i}, \quad \text{and} \quad \zeta_i(u) \neq \zeta_j(u) \quad \text{for} \quad i \neq j.$$

   In this case the stationary points of $S(u,\zeta)$ are non-degenerate and correspond to the roots of the function $Q(u,\xi)$ with absolute values equal to 1.

4. If $u \in \mathbb{C}\backslash\mathbb{U}$, then

$$\zeta_0(u) = (1+\omega)e^{i\varphi/2}, \quad \zeta_1(u) = e^{-i\varphi}, \quad \zeta_2(u) = (1+\omega)^{-1}e^{i\varphi/2}$$

for certain φ and $\omega > 0$.

In this case the stationary points of the function $S(u,\zeta)$ are non-degenerate, and correspond to the roots of the function $Q(u,\xi)$ that can be expressed as $\xi_0 = (1+\tau)e^{i\varphi}$, $\xi_1 = e^{-2i\varphi}$, $\xi_2 = (1+\tau)^{-1}e^{i\varphi}$, $(1+\tau) = (1+\omega)^2$.

Lemma 3.1 is proved in section 5.

Formula (3.8) and Lemma 3.1 give a complete description of the stationary points of the function $S(u,\zeta)$.

In order to estimate the large–time behavior of the integral having the form

$$I(t,u,\lambda) = \iint\limits_{\mathbb{C}} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta \tag{3.9}$$

uniformly on $u, \lambda \in \mathbb{C}$, in the present and the following sections we will use the following general scheme.

1. Consider D_ε, the union of disks with a radius of ε and centers in singular points of function $f(\zeta,\lambda)$ and stationary points of $S(u,\zeta)$.

2. Represent $I(t,u,\lambda)$ as the sum of integrals over D_ε and $\mathbb{C}\backslash D_\varepsilon$:

$$\begin{aligned} &I(t,u,\lambda) = I_{int} + I_{ext}, \qquad \text{where} \\ &I_{int} = \iint\limits_{D_\varepsilon} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \\ &I_{ext} = \iint\limits_{\mathbb{C}\backslash D_\varepsilon} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \end{aligned} \tag{3.10}$$

3. Find an estimate of the form

$$|I_{int}| = O\left(\varepsilon^\alpha\right), \quad \text{as} \quad \varepsilon \to 0 \quad (\alpha \geq 1)$$

uniformly on u, λ, t.

4. Integrate I_{ext} by parts using Stokes formula

$$I_{ext} = \frac{1}{2t}\int\limits_{\partial D_\varepsilon} \frac{f(\zeta,\lambda)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar{\zeta} - \frac{1}{it}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f'_\zeta(\zeta,\lambda)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta -$$
$$- \frac{1}{it}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f(\zeta,\lambda)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$
$$= -\frac{1}{t}(I_1 - I_2 - I_3). \quad (3.11)$$

5. For each I_i find an estimate of the form

$$(a)\ |I_i| = O\left(\ln\frac{1}{\varepsilon}\right) \quad \text{or} \quad (b)\ |I_i| = O\left(\frac{1}{\varepsilon^\beta}\right), \qquad \text{as } \varepsilon \to 0.$$

6. In case (a) set $\varepsilon = \dfrac{1}{|t|}$ which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \text{as} \quad t \to \infty.$$

   In case (b) set $\varepsilon = \dfrac{1}{|t|^k}$, where $k(\alpha+\beta) = 1$, which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{1}{|t|^{\frac{\alpha}{\alpha+\beta}}}\right), \quad \text{as} \quad t \to \infty.$$

Using this scheme we obtain, in particular, the following result

Lemma 3.2. *Under assumptions (3.2), (3.4),*

$$|I(t,u)| = O\left(\frac{\ln(3+|t|)}{1+|t|}\right) \quad \textit{for} \quad t \in \mathbb{R}$$

uniformly on $u \in \mathbb{C}$.

A detailed proof of Lemma 3.2 is given in section 5.

4 Estimate for the non–linearized case

In this section we prove estimate (1.14) for the solution $v(x,t)$ of the Cauchy problem for the Novikov–Veselov equation at positive energy with the initial data $v(x,0)$ satisfying properties (1.10)–(1.13) or, more generally, for $v(x,t)$ constructed by means of (2.6)–(2.13).

We proceed from the formulas (2.12), (2.13) for the potential $v(z,t)$ and the integral equation (2.11) for $\mu(z,\lambda,t)$.

We write (2.11) as

$$\mu(z,\lambda,t) = 1 + (A_{z,t}\mu)(z,\lambda,t), \tag{4.1}$$

where

$$(A_{z,t}f)(\lambda) = \partial_{\bar\lambda}^{-1}(r(\lambda)\exp(itS(u,\lambda))\overline{f(\lambda)}) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{f(\zeta)}}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and $S(u,\zeta)$ is defined by (3.5), $u = \dfrac{z}{t}$.

Equation (4.1) can be also written in the form

$$\mu(z,\lambda,t) = 1 + A_{z,t}\cdot 1 + (A^2_{z,t}\mu)(z,\lambda,t). \tag{4.2}$$

According to the theory of the generalized analytic functions (see [14]), equations (4.1), (4.2) have a unique solution for all z,t. This solution can be written as

$$\mu(z,\lambda,t) = (I - A^2_{z,t})^{-1}(1 + A_{z,t}\cdot 1). \tag{4.3}$$

Equation (4.3) possesses a formal asymptotic expansion

$$\mu(z,\lambda,t) = (I + A^2_{z,t} + A^4_{z,t} + \ldots)(1 + A_{z,t}\cdot 1). \tag{4.4}$$

From estimate (4.10) given below it follows that (4.4) uniformly converges for sufficiently large t. We will also write formula (4.4) in the form

$$\mu(z,\lambda,t) = 1 + A_{z,t}\cdot 1 + R, \tag{4.5}$$

where $R = \left(\sum\limits_{k=1}^{\infty} A^{2k}_{z,t}\right)(1 + A_{z,t}\cdot 1)$.

In addition to $A_{z,t}$ we introduce another integral operator $B_{z,t}$ defined as

$$B_{z,t}\cdot f = \iint\limits_{\mathbb{C}} r(\zeta)\exp(itS(u,\zeta))\overline{f(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{4.6}$$

To study (4.4) we will need some estimates on the values of operators $A_{z,t}$ and $B_{z,t}$.

Lemma 4.1. *Under assumptions (2.8)–(2.10), the following estimates hold:*

(a)

$$|B_{z,t} \cdot 1| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{4.7}$$

uniformly on $u \in \mathbb{C}$*;*

(b)

$$|B_{z,t} \cdot A_{z,t} \cdot 1| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{4.8}$$

uniformly on $u \in \mathbb{C}$*;*

(c)

$$|(A_{z,t}^2 \cdot 1)(\lambda)| \leqslant \frac{\beta}{|t|^{1/14}}, \quad |t| \geqslant 1, \tag{4.9}$$

where β *is a constant independent of* u *and* λ*;*

(d)

$$|(A_{z,t}^n \cdot 1)(\lambda)| \leqslant \frac{\beta^{n-1}}{|t|^{\lfloor n/2 \rfloor/14}}, \quad |t| \geqslant 1; \tag{4.10}$$

(e)

$$|B_{z,t} \cdot A_{z,t}^{n-1} \cdot 1| \leqslant \frac{\beta^{n-1}\ln(3+|t|)}{|t|^{1+\lfloor (n-2)/2 \rfloor/14}}, \quad |t| \geqslant 1. \tag{4.11}$$

Proof of Lemma 4.1. We proceed according to the scheme described in the previous section.

(a) This point follows from Lemma 3.2.

(b) As $(A_{z,t} \cdot 1)(\lambda) \in C(\mathbb{C})$, we take D_ε to be the union of disks of a radius ε centered in the stationary points of $S(u,\zeta)$. We note that $(A_{z,t} \cdot 1)(\lambda) = O(1)$ uniformly on u (or, equivalently, on z) and λ. Thus the integral I_{int} (as in (3.10)) can be estimated as

$$I_{int} = \iint\limits_{D_\varepsilon} r(\zeta,\bar{\zeta})\exp(itS(u,\zeta,\bar{\zeta}))(A_{z,t} \cdot 1)(\zeta)d\mathrm{Re}\zeta d\mathrm{Im}\zeta = O(\varepsilon^2).$$

Now let us estimate the integral I_{ext} (as in (3.10)). For this purpose we apply the Stokes formula (as in (3.11)) taking into consideration that

$\partial_{\bar{\lambda}}(A_{z,t} \cdot f)(\lambda) = r(\lambda)\exp(itS(u,\lambda))\overline{f(\lambda)}$:

$$\iint\limits_{\mathbb{C}\backslash D_\varepsilon} r(\zeta)\exp(itS(u,\zeta))(A_{z,t}\cdot 1)(\zeta)d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \int\limits_{\partial D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))(A_{z,t}\cdot 1)(\zeta)}{2tS'_{\bar{\zeta}}(u,\zeta)} d\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r'_{\bar{\zeta}}(\zeta)\exp(itS(u,\zeta))(A_{z,t}\cdot 1)(\zeta)}{itS'_{\bar{\zeta}}(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))(A_{z,t}\cdot 1)(\zeta)S''_{\bar{\zeta}\bar{\zeta}}(u,\zeta)}{it(S'_{\bar{\zeta}}(u,\zeta))^2} d\mathrm{Re}\zeta d\mathrm{Im}\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r^2(\zeta)(\exp(itS(u,\zeta)))^2}{itS'_{\bar{\zeta}}(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \quad (4.12)$$

Now, proceeding as in the proof of Lemma 3.2, we obtain (4.8).

(c) In this case we build D_ε as the union of disks with a radius of ε and centers in λ and stationary points of $S(u,\zeta)$. The integral I_{int} over D_ε behaves asymptotically as $O(\varepsilon)$. When estimating the integral I_{ext} over $\mathbb{C}\backslash D_\varepsilon$ we use (3.8), (3.11) and the following inequalities

$$|\zeta - \lambda| \geqslant \varepsilon, \quad |\zeta - \zeta_i| \geqslant \varepsilon$$

(ζ_i are stationary points of $S(u,\zeta)$) which hold for all $\zeta \in \mathbb{C}\backslash D_\varepsilon$. Thus we obtain that the asymptotical behavior of I_{ext} is at most $O\left(\frac{1}{|t|\varepsilon^{13}}\right)$. Then, as proposed by the scheme, we choose $\varepsilon = |t|^{-1/14}$ and obtain the required estimate.

(d) This point is proved by induction. As in point *(c)* D_ε is the union of disks with a radius of ε and centers in λ and stationary points of $S(u,\zeta)$.

For the integral I_{int} we have

$$|I_{int}| = \left| \iint\limits_{D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))}{\zeta-\lambda}(A_{z,t}^{n-1}\cdot 1)(\zeta)d\mathrm{Re}\zeta d\mathrm{Im}\zeta \right| \leqslant$$

$$\leqslant \frac{\beta^{n-2}}{|t|^{\lfloor (n-1)/2\rfloor/14}} \iint\limits_{D_\varepsilon} \left| \frac{r(\zeta)\exp(itS(u,\zeta))}{\zeta-\lambda} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\beta^{n-1}\varepsilon}{|t|^{\lfloor (n-2)/2\rfloor/14}}. \tag{4.13}$$

To estimate I_{ext} we use the following representation

$$\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))(A_{z,t}^{n-1}\cdot 1)(\zeta)}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \int\limits_{\partial D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))(A_{z,t}^{n-1}\cdot 1)(\zeta)}{2t(\zeta-\lambda)S'_{\bar\zeta}(u,\zeta)} d\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r'_{\bar\zeta}(\zeta)\exp(itS(u,\zeta))(A_{z,t}^{n-1}\cdot 1)(\zeta)}{it(\zeta-\lambda)S'_{\bar\zeta}(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))(A_{z,t}^{n-1}\cdot 1)(\zeta)S''_{\bar\zeta\bar\zeta}(u,\zeta)}{it(\zeta-\lambda)(S'_{\bar\zeta}(u,\zeta))^2} d\mathrm{Re}\zeta d\mathrm{Im}\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r^2(\zeta)(\exp(itS(u,\zeta)))^2(A_{z,t}^{n-2}\cdot 1)(\zeta)}{it(\zeta-\lambda)S'_{\bar\zeta}(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= J_1 + J_2 + J_3 + J_4. \tag{4.14}$$

The integrals J_i can be estimated in the following way

$$|J_1| \leqslant \frac{\beta^{n-2}}{|t|\cdot|t|^{\lfloor (n-1)/2\rfloor/14}} \frac{1}{\varepsilon^7} \int\limits_{\partial D_\varepsilon} |r(\zeta)|d\zeta \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor (n-1)/2\rfloor/14}} \frac{1}{\varepsilon^7}.$$

Similarly,

$$|J_2| \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor (n-1)/2\rfloor/14}} \frac{1}{\varepsilon^7},$$

$$|J_3| \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor (n-1)/2\rfloor/14}} \frac{1}{\varepsilon^{13}},$$

$$|J_4| \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor (n-2)/2\rfloor/14}} \frac{1}{\varepsilon^7}.$$

Thus,

$$|I_{ext}| \leqslant \frac{\beta^{n-1}}{|t| \cdot |t|^{\lfloor (n-2)/2 \rfloor /14} \cdot \varepsilon^{13}}.$$

Now we set $\varepsilon = |t|^{-1/14}$ and obtain the overall estimate

$$|(A^n_{z,t} \cdot 1)(\lambda)| \leqslant \frac{\beta^{n-1}}{|t|^{1/14}|t|^{\lfloor (n-2)/2 \rfloor /14}} = \frac{\beta^{n-1}}{t^{\lfloor n/2 \rfloor /14}}.$$

(e) Proceeding from *(d)* this point is proved similarly to *(b)*. □

Lemma 4.2. *Under the assumptions of Lemma 4.1, we have that:*

(a) $A_{z,t} \cdot 1 = \frac{a_1(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *for* $\lambda \to \infty$, *where*

$$|\partial_z a_1(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{4.15}$$

uniformly on $z \in \mathbb{C}$.

(b) $A^2_{z,t} \cdot 1 = \frac{a_2(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *for* $\lambda \to \infty$, *where*

$$|\partial_z a_2(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{4.16}$$

uniformly on $z \in \mathbb{C}$.

(c) $R = \frac{q(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *as* $\lambda \to \infty$, *where*

$$|\partial_z q(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{4.17}$$

uniformly on $z \in \mathbb{C}$.

Proof of Lemma 4.2. The asymptotics for $A_{z,t} \cdot 1$, $A^2_{z,t} \cdot 1$ and R follow from the definitions of $A_{z,t}$ and R, formula (2.12) and properties (2.8), (2.9). The rest of the proof consists in the following.

(a) Estimate (4.15) follows from (4.7) and the formula

$$a_1(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta) \exp(itS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = B_{z,t} \cdot 1.$$

(b) Estimate (4.16) follows from (4.8) and the formula

$$a_2(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta,\bar\zeta) \exp(itS(u,\zeta,\bar\zeta))(A_{z,t}\cdot 1)(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = B_{z,t}\cdot A_{z,t}\cdot 1.$$

(c) We note that $q(z,t) = \sum\limits_{k=2}^{\infty} a_k(z,t)$ where $a_k(z,t)$ is defined

$$(A^k_{z,t}\cdot 1)(\lambda) = \frac{a_k(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right).$$

Next,

$$a_k(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta) \exp(itS(u,\zeta))(A^{k-1}_{z,t}\cdot 1)(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta.$$

Thus, proceeding as in point *(e)* of lemma 4.1, we obtain that $a_2(z,t) + a_3(z,t) = O\left(\frac{\ln(|t|)}{|t|}\right)$ and the rest of the members form a geometric progression that converges to the sum of order $O\left(\frac{\ln(|t|)}{|t|^{1+1/14}}\right)$. $\square$

Formulas (2.12), (2.13), (4.5) and Lemma 4.2 imply

Theorem 4.1. *Let $v(x,t)$ be a solution to the Cauchy problem for the Novikov–Veselov equation (1.6) with $E = 1$, constructed via (2.11)–(2.13) under assumptions (2.6)–(2.10). Then*

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v)\ln(3+|t|)}{1+|t|}, \quad x\in\mathbb{R}^2, t\in\mathbb{R}.$$

5 Proofs of Lemmas 3.1 and 3.2

Proof of Lemma 3.1. Under the additional assumption that $\xi \neq 0$ the system of equations (3.6)–(3.7) is equivalent to the following system

$$\begin{cases} \xi^3 - \frac{\bar u}{6}\xi^2 + \frac{u}{6}\xi - 1 = 0, \\ \xi^3 - \frac{u}{6}\xi + 2 = 0. \end{cases} \tag{5.1}$$

We claim that $\xi = \xi_1$ corresponds to a degenerate stationary point of (3.5), iff

the polynomial $P(\xi) = \xi^3 - \dfrac{\bar u}{6}\xi^2 + \dfrac{u}{6}\xi - 1$ can be represented in the form $P(\xi) = (\xi-\xi_1)^2(\xi-\xi_2)$. (5.2)

Indeed, if $\xi = \xi_1 \neq 0$ is a zero of the function $Q(u,\xi) = -\frac{\bar{u}}{2} + \frac{u}{2\xi} + 3\xi - \frac{3}{\xi^2} = 3P(\xi)/\xi^2$, then $Q(u,\xi)$ is holomorphic in a certain neighborhood of $\xi = \xi_1$ and can be expanded into the following Taylor series

$$Q(u,\xi) = c_1(\xi - \xi_1) + c_2(\xi - \xi_1)^2 + c_3(\xi - \xi_1)^3 + \dots$$

Thus S'_ζ can be represented as

$$S'_\zeta(u,\zeta) = c_1(\zeta^2 - \xi_1) + c_2(\zeta^2 - \xi_1)^2 + c_3(\zeta^2 - \xi_1)^3 + \dots$$

After differentiating with respect to ζ we obtain

$$S''_{\zeta\zeta}(u,\zeta) = 2c_1\zeta + 4c_2(\zeta^2 - \xi_1)\zeta + 6c_3(\zeta^2 - \xi_1)^2\zeta + \dots$$

The stationary point corresponding to $\xi = \xi_1$ can be degenerate if and only if $c_1 = 0$.

So for the polynomial $P(\xi)$ we get the following representation in the neighborhood of $\xi = \xi_1$

$$P(\xi) = c_2(\xi - \xi_1)^2\xi^2 + O(|\xi - \xi_1|^3).$$

As $P(\xi)$ is a third–order polynomial, it follows that $P(\xi) = (\xi - \xi_1)^2(\xi - \xi_2)$.

Expanding the expression for $P(\xi)$ and equating coefficients for the corresponding powers of ξ results in the following system

$$\begin{cases} 2\xi_1 + \xi_2 = \frac{\bar{u}}{6}, \\ 2\xi_1\xi_2 + \xi_1^2 = \frac{u}{6}, \\ \xi_1^2\xi_2 = 1. \end{cases} \tag{5.3}$$

Excluding u from the system yields

$$\begin{cases} 2\xi_1\xi_2 + \xi_1^2 = 2\bar{\xi}_1 + \bar{\xi}_2, \\ \xi_1^2\xi_2 = 1. \end{cases} \tag{5.4}$$

We represent ξ_1 in the form $\xi_1 = re^{i\varphi}$. Then from the second equation in (5.4) we get that $\xi_2 = r^{-2}e^{-2i\varphi}$. Substituting this into the first equation of (5.4) yields

$$(r^2 - r^{-2})e^{2i\varphi} = (r - r^{-1})e^{-i\varphi}. \tag{5.5}$$

For $r = 1$ equation (5.5) holds for all φ. In addition, if $r \neq 1$, $r > 0$, then equation (5.5) can be rewritten as

$$(r + r^{-1})e^{3i\varphi} = 2$$

and has no solutions for real φ.

Using now the second equation in (5.3) one can see that the set of u values, for which (5.4), (5.3) are solvable and (5.2) holds, is a curve on the complex plane described in the parametric form by

$$u = 6(2e^{-i\varphi} + e^{2i\varphi})$$

(see Figure 1). This curve has three singular points corresponding to $\varphi = \frac{2\pi k}{3}$, $k = 0, 1, 2$. For these values of φ we have that $e^{i\varphi} = \xi_1 = \xi_2 = e^{-2i\varphi}$, and $P(\xi)$ can be represented in the form $P(\xi) = (\xi - \xi_1)^3$. For $\varphi \neq \frac{2\pi k}{3}$, $k = 0, 1, 2$, we have that $P(\xi) = (\xi - \xi_1)^2(\xi - \xi_2)$ where $e^{i\varphi} = \xi_1 \neq \xi_2 = e^{-2i\varphi}$. The first two statements of the Lemma 3.1 are proved.

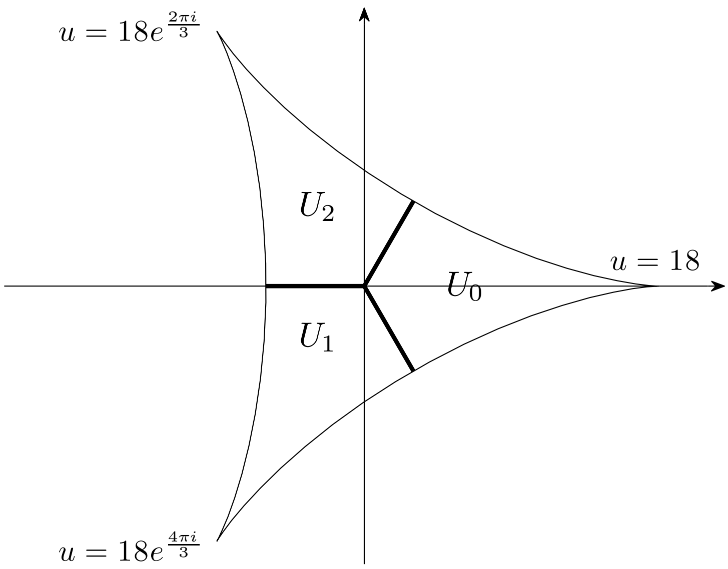

Figure 1: The curve $\mathcal{U}$ on the complex plane.

Let us now fix $\xi = e^{i\varphi}$ and find the set of u for which this ξ is a root of the polynomial $P(\xi)$.

One can see that ξ is the root of $P(\xi)$ iff

$$\xi\frac{u}{6} - \xi^2\frac{\bar{u}}{6} = 1 - \xi^3. \tag{5.6}$$

We now solve the homogeneous equation

$$e^{i\varphi}\frac{u}{6} - e^{2i\varphi}\frac{\bar{u}}{6} = 0 \tag{5.7}$$

with respect to u to find the plausible perturbations of u for which $\xi = e^{i\varphi}$ remains a root of $P(\xi)$.

From (5.7) we get $\frac{u}{\bar{u}} = e^{i\varphi}$, and thus $u = se^{i\varphi/2}$ for $s \in \mathbb{R}$. So for all u that belong to the line

$$u(\varphi, s) = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R} \tag{5.8}$$

one of the roots of $P(\xi)$ is equal to $e^{i\varphi}$.

Now we note that the tangent vector to $\mathcal{U}$

$$\frac{d}{d\varphi}(2e^{-i\varphi} + e^{2i\varphi}) = 2i(e^{2i\varphi} - e^{-i\varphi})$$

is collinear to the perturbation vector $se^{i\varphi/2}$ for all $\varphi \neq \frac{2\pi ik}{3}$, $k = 0, 1, 2$. Thus $u(\varphi, s)$ given by (5.8) is the tangent line to $\mathcal{U}$ passing through the point $6(2e^{-i\varphi} + e^{2i\varphi})$.

We note that for each $u \in \mathrm{int}\mathbb{U}$ there exist two different tangents to the curve $\mathcal{U}$ passing through this u. Indeed, note that the tangent lines to the curve $\mathcal{U}$ passing through the points $u = 18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, divide the domain $\mathrm{int}\mathbb{U}$ into three parts

$$\begin{aligned}
U_0 &= \{u = re^{i\varphi},\ 0 \leqslant r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \frac{-\pi}{3} \leq \varphi \leq \frac{\pi}{3}\},\\
U_1 &= \{u = re^{i\varphi},\ 0 < r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \frac{\pi}{3} < \varphi \leq \pi\},\\
U_2 &= \{u = re^{i\varphi},\ 0 < r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \pi < \varphi < \frac{5\pi}{3}\}
\end{aligned}$$

(see Figure 1). We first study U_0. Let us consider K_{01} and K_{02} which are the sets of points of the following pencils of tangent lines:

$$\begin{aligned}
K_{01} &= \left\{u \in \mathbb{C} \colon u = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R}, \quad -\frac{2\pi}{3} \leqslant \varphi < 0\right\},\\
K_{02} &= \left\{u \in \mathbb{C} \colon u = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R}, \quad 0 < \varphi \leqslant \frac{2\pi}{3}\right\}.
\end{aligned}$$

It is easily seen that $U_0 \subset K_{01}$, $U_0 \subset K_{02}$, i.e. each point from U_0 is covered by a certain tangent line from both pencils K_{01} and K_{02}. It can be shown similarly that each point of U_j, $j = 1, 2$, is covered twice by the corresponding tangent lines.

Thus every point from $\mathrm{int}\mathbb{U}$ is covered twice which means that for each u in the domain limited by the curve $\mathcal{U}$ there exist two different roots of $P(\xi)$ equal to 1 in absolute value. As the product of the roots of the polynomial

$P(\xi)$ is equal to 1, the third root of the polynomial is also equal to 1 in absolute value. We do not consider the values u from the boundary of $\mathbb{U}$ which means that the above–mentioned roots correspond to non–degenerate stationary points. Thus the third statement of Lemma 3.1 is proved.

Now suppose $u \in \mathbb{C}\backslash\mathbb{U}$. We note that every such point u belongs to one and only one tangent line to the curve $\mathcal{U}$. That means that for every $u \in \mathbb{C}\backslash\mathbb{U}$ one of the roots of the polynomial $P(\xi)$, ξ_1, is such that $|\xi_1| = 1$. As $u \notin \mathcal{U}$, $\xi_0 \neq \xi_1$, $\xi_2 \neq \xi_1$. Besides $|\xi_0| \neq 1$ because otherwise there would be two different tangent lines passing through the corresponding point u. That means that for every $u \in \mathbb{C}\backslash\mathbb{U}$ there exists a root of the polynomial $P(\xi)$, namely $\xi = \xi_0$, such that $|\xi_0| \neq 1$.

Considering equation (5.6) and its conjugate yields the following system of linear equations for u and $\bar{u}$

$$\begin{cases} au + b\bar{u} = c, \\ \bar{b}\bar{u} + \bar{a}\bar{u} = \bar{c}, \end{cases}$$

where $a = \frac{\xi}{6}$, $b = -\frac{\xi^2}{6}$, $c = 1 - \xi^3$ for each $\xi \in \mathbb{C}$, $|\xi| \neq 1$, $\xi \neq 0$. Thus

$$u = \frac{c\bar{a} - \bar{c}b}{a\bar{a} - b\bar{b}} = 6\frac{\bar{\xi} - \bar{\xi}\xi^3 + \xi^2 - \xi^2\bar{\xi}^3}{\xi\bar{\xi}(1 - \xi\bar{\xi})} = 6\left(\frac{1}{\xi} + \bar{\xi} + \frac{\xi}{\bar{\xi}}\right)$$

for each $\xi \in \mathbb{C}$, $|\xi| \neq 1$, $\xi \neq 0$.

Now let us consider $\xi = \xi_0 = (1 + \tau)e^{i\varphi}$, $0 < \tau < +\infty$. Then the corresponding value of the parameter u is

$$u = 6\left(2e^{-i\varphi} + e^{2i\varphi} + \frac{\tau^2}{1 + \tau}e^{-i\varphi}\right).$$

In addition to $\xi = \xi_0$, for this value of the parameter u polynomial $P(\xi)$ has also a root $\xi = \xi_1 = e^{-2i\varphi}$. Indeed, if $u = 6(2e^{-i\varphi} + e^{2i\varphi})$, then $\xi_1 = e^{-2i\varphi}$ is a root of the polynomial $P(\xi)$, and the plausible perturbation of this u for which ξ_1 remains a root of $P(\xi)$ is equal to $se^{-i\varphi}$.

In addition, as the product of the roots of P is equal to 1, the third root is $\xi_2 = (1 + \tau)^{-1}e^{i\varphi}$. The fourth statement of Lemma 3.1 is proved. $\square$

Proof of lemma 3.2. In this case D_ε is the union of disks with a radius of ε centered in the stationary points of $S(u, \zeta)$. The integral I_{int} (as in (3.10)) is estimated as

$$|I_{int}| = \left|\iint\limits_{D_\varepsilon} r(\zeta)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| \leqslant \mathrm{const}\left|\iint\limits_{D_\varepsilon} d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| = O(\varepsilon^2).$$

The estimate for I_{ext} (as in (3.10)) is proved separately for $u \in \mathbb{U}$ and $u \in \mathbb{C}\backslash\mathbb{U}$.

I. $u \in \mathbb{U}$:

In this case all $\zeta \in D_\varepsilon$ lie in the ε–neighborhood of the unit circle. Consequently, from (2.6), (2.8) and (2.10) it follows that for any $N \in \mathbb{N}$ and any $\varepsilon_0 \in (0, 1/2]$ there exists $C = C(\varepsilon_0, N)$ such that

$$|r(\zeta)| \leqslant C\rho^N, \quad |r'_\zeta(\zeta)| \leqslant C\rho^N$$

for all $\zeta \in D_\rho$, $\rho \leqslant \varepsilon_0$.

The function $S'_\zeta(u,\zeta)$ can be estimated as

$$\begin{aligned} &|S'_\zeta(u,\zeta)| \geqslant 3\frac{\varepsilon_0^6}{|\zeta|^4} \quad \text{for}\, \zeta \in \mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and} \\ &|S'_\zeta(u,\zeta)| \geqslant 3\frac{\rho^6}{|\zeta|^4} \quad \text{for}\, \zeta \in \partial D_\rho, \quad \varepsilon \leqslant \rho \leqslant \varepsilon_0. \end{aligned}$$

Taking $N = 5$ results in the following estimate for I_1

$$\begin{aligned} |I_1| \leqslant \frac{1}{2}\int\limits_{\partial D_\varepsilon} \frac{|r(\zeta)|}{|S'_\zeta(u,\zeta)|} d\bar\zeta \leqslant \text{const}\frac{\varepsilon^N}{\varepsilon^6}\int\limits_{\partial D_\varepsilon} |\zeta|^4 d\bar\zeta \leqslant \\ \leqslant \text{const}\frac{\varepsilon^N\varepsilon}{\varepsilon^6}(1+\varepsilon)^4 = O(1), \quad \text{as}\, \varepsilon \to 0. \end{aligned}$$

When estimating I_2 and I_3 we integrate separately over $D_{\varepsilon_0}\backslash D_\varepsilon$ and $\mathbb{C}\backslash D_\varepsilon$:

$$\begin{aligned} |I_2| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon} \left|\frac{r'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}} \left|\frac{r'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \\ \leqslant \text{const}\int\limits_\varepsilon^{\varepsilon_0} \frac{\rho^N\rho}{\rho^6} d\rho + \text{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}} |r(\zeta)||\zeta^4| d\mathrm{Re}\zeta d\mathrm{Im}\zeta = O(1), \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

$$|I_3| \leqslant \iint\limits_{D_{\varepsilon_0}\setminus D_\varepsilon} \left| \frac{r(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\setminus D_{\varepsilon_0}} \left| \frac{r(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant$$

$$\leqslant \mathrm{const} \int\limits_{\varepsilon}^{\varepsilon_0} \frac{\rho^N \rho}{\rho^{12}} d\rho + \mathrm{const} \iint\limits_{\mathbb{C}\setminus D_{\varepsilon_0}} |r(\zeta)||\zeta^3| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \overset{N=11}{=} O(1), \quad \text{as} \quad \varepsilon \to 0.$$

Setting finally $\varepsilon = \frac{1}{|t|}$ yields

$$I(t,u) = O\left(\frac{1}{|t|}\right), \quad \text{as} \quad t \to \infty$$

uniformly on $u \in \mathbb{U}$.

II. $u \in \mathbb{C}\setminus\mathbb{U}$:

Let us divide the complex plane into six sets, each containing one and only one stationary point of $S(u,\zeta)$: $\mathbb{C} = \bigcup\limits_{k=0}^{2} \left(Z_k^+ \bigcup Z_k^-\right)$. We define the set $Z_k^\pm$ as the set of points of the complex plane to which the stationary point $\pm\zeta_k$ is the closest:

$$\begin{aligned} Z_k^+ &= \{\zeta \in \mathbb{C} \colon |\zeta - \zeta_i| \geqslant |\zeta - \zeta_k|, \quad |\zeta + \zeta_j| \geqslant |\zeta - \zeta_k|, \quad i,j \in \{0,1,2\}\}, \\ Z_k^- &= \{\zeta \in \mathbb{C} \colon |\zeta - \zeta_i| \geqslant |\zeta + \zeta_k|, \quad |\zeta + \zeta_j| \geqslant |\zeta + \zeta_k|, \quad i,j \in \{0,1,2\}\} \end{aligned}$$

where $k \in \{0,1,2\}$. We will estimate the integral over each $\mathcal{Z}_k^+$ separately. The integrals over Z_k^- are treated similarly.

Let us first take $\zeta \in Z_1^+$. Using the definition of Z_1^+ and the property that all $\zeta \in D_\varepsilon \bigcap Z_1^+$ lie in the ε–neighborhood of the unit circle one can see that the scheme of reasoning for the case I is applicable.

Now let us consider $\zeta \in Z_0^+ \bigcup Z_2^+$.

(A) First, we will study the set of values of parameter ω for which $\zeta_2(u)$, $-\zeta_2(u)$ lie outside the $2\varepsilon_0$–neighborhood of zero, i.e. $\frac{1}{1+\omega} > 2\varepsilon_0$.

We will consider $\zeta \in Z_0^+$ (the case $\zeta \in Z_2^+$ is treated similarly). If $\zeta \in Z_0^+ \bigcap D_{\varepsilon_0}$ (for a certain ε_0), then it can be represented as

$$\zeta = (1+\omega)e^{i\varphi/2} + \rho e^{i\theta}, \quad \rho \leqslant \varepsilon_0.$$

Let us estimate the ratio $\dfrac{r(\zeta)}{S'_\zeta(\zeta,u)}$ in D_{ε_0}. If ζ_0 belongs to the $2\varepsilon_0$–neighborhood of ζ_1, then all $\zeta\in Z_0^+\bigcap D_{\varepsilon_0}$ belong to the $3\varepsilon_0$–neighborhood of ζ_1. The following estimates hold:

$$|\zeta+\zeta_1|\geqslant 2-3\varepsilon_0,\quad |\zeta+\zeta_0|>2-\varepsilon_0,\quad |\zeta+\zeta_2|\geqslant 1+\varepsilon_0,\quad \xi\in Z_0^+\bigcap D_{\varepsilon_0}. \tag{5.9}$$

Further, we note that for any $N\in\mathbb{N}$ there exists a function $\tilde r(\zeta)$ that can be represented in the form (2.6) with a certain $\tilde b(\zeta)$ satisfying properties (2.8)–(2.10), such that $|r(\zeta)|\leqslant|\zeta-\zeta_1|^N|\tilde r(\zeta)|$ for ζ belonging to the $3\varepsilon_0$–neighborhood of ζ_1. This and (5.9) imply that

$$\left|\frac{r(\zeta)}{S'_\zeta(\zeta,u)}\right|\leqslant \mathrm{const}\frac{|\tilde r(\zeta)||\zeta|^4}{|\zeta-\zeta_2||\zeta-\zeta_0|}. \tag{5.10}$$

A similar reasoning holds for the case when ζ_0 belongs to the $2\varepsilon_0$–neighborhood of $-\zeta_1$. Now if ζ_0 does not belong to the $2\varepsilon_0$–neighborhood of ζ_1 and $-\zeta_1$, then $|\zeta-\zeta_1|\geqslant\varepsilon_0$ and $|\zeta+\zeta_1|\geqslant\varepsilon_0$ for all $\zeta\in Z_0^+\bigcap D_{\varepsilon_0}$. Two last estimates of (5.9) hold and thus (5.10) holds with $\tilde r(\zeta)=r(\zeta)$.

The difference $|\zeta-\zeta_2|$ can be estimated

$$|\zeta-\zeta_2|\geqslant\frac{1}{2}|\zeta_0-\zeta_2|=\frac{\omega(2+\omega)}{2(1+\omega)},\quad \zeta\in Z_0^+. \tag{5.11}$$

In order to get rid of this member in the denominator, let us represent $\tilde r(\zeta)$ by the Taylor formula in the neighborhood of $(1+\omega)e^{i\varphi/2}$:

$$\tilde r(\zeta)=\tilde r(e^{i\varphi/2}+\omega e^{i\varphi/2})+\tilde r'(e^{i\varphi/2}+\omega e^{i\varphi/2}+se^{i\theta})\rho,\quad s=\lambda\rho \text{ for some } \lambda\in[0,1],$$

where $'$ denotes the derivative with respect to s and where λ depends, in particular, on ρ.

For an arbitrary value of ω the following estimates hold:

$$|\tilde r(e^{i\varphi/2}+\omega e^{i\varphi/2})|\leqslant\mathrm{const}|\omega|^N, \tag{5.12}$$

$$|\tilde r'(e^{i\varphi/2}+\omega e^{i\varphi/2})|\leqslant\mathrm{const}|\omega|^N. \tag{5.13}$$

This finally yields

$$\frac{|\tilde r(\zeta)||\zeta|^4}{|\zeta-\zeta_2||\zeta-\zeta_0|}\leqslant\mathrm{const}\left(\frac{|\tilde r(e^{i\varphi/2}+\omega e^{i\varphi/2})|}{\frac{\omega(2+\omega)}{2(1+\omega)}|\zeta-\zeta_0|}+\frac{|\tilde r'(e^{i\varphi/2}+\omega e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho}{\rho|\zeta-\zeta_0|}\right)\leqslant$$

$$\leqslant\frac{\mathrm{const}}{|\zeta-\zeta_0|},\quad \zeta\in Z_0^+\bigcap D_{\varepsilon_0}.$$

Now we are ready to estimate I_1:

$$\int\limits_{Z_0^+\bigcap\partial D_\varepsilon}\left|\frac{r(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right|d\bar{\zeta}\leqslant \mathrm{const}\int\limits_{Z_0^+\bigcap\partial D_\varepsilon}\frac{d\bar{\zeta}}{|\zeta-\zeta_0|}=O\,(1)\quad\text{as}\quad\varepsilon\to 0.$$

For I_2 we note that the estimate

$$\left|\frac{r'_\zeta(\zeta)}{S'_\zeta(u,\zeta)}\right|\leqslant\frac{\mathrm{const}}{|\zeta-\zeta_0|},\quad \zeta\in Z_0^+\bigcap D_{\varepsilon_0},$$

can be obtained using the same reasoning as for the ratio $\dfrac{r(\zeta)}{S'_\zeta(u,\zeta)}$.
Thus

$$\iint\limits_{Z_0^+\backslash D_\varepsilon}\left|\frac{r'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right|d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta\leqslant \mathrm{const}\left(\iint\limits_{Z_0^+\backslash D_{\varepsilon_0}}|r'_\zeta(\zeta)||\zeta|^4 d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta+\right.$$

$$\left.+\iint\limits_{Z_0^+\bigcap(D_{\varepsilon_0}\backslash D_\varepsilon)}\frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{|\zeta-\zeta_0|}\right)=O(1)\quad\text{as}\quad\varepsilon\to 0.$$

In order to estimate $\dfrac{r(\zeta)}{(S'_\zeta(u,\zeta))^2}$ in D_{ε_0} we take the members in the Taylor formula for $\tilde{r}(\zeta)$ up to the second order:

$$\tilde{r}(\zeta)=\tilde{r}(e^{i\varphi/2}+\omega e^{i\varphi/2})+\tilde{r}'(e^{i\varphi/2}+\omega e^{i\varphi/2})\rho+\frac{1}{2}\tilde{r}''(e^{i\varphi/2}+\omega e^{i\varphi/2}+\lambda\rho e^{i\theta})\rho^2. \tag{5.14}$$

From formulas (5.10)–(5.14) it follows that

$$\left|\frac{r(\zeta)}{(S'_\zeta(u,\zeta))^2}\right|\leqslant\frac{|\tilde{r}(\zeta)||\zeta|^8}{|\zeta-\zeta_2|^2|\zeta-\zeta_0|^2}\leqslant \mathrm{const}\left(\frac{|\tilde{r}(e^{i\varphi/2}+\omega e^{i\varphi/2})|}{\left(\frac{\omega(2+\omega)}{2(1+\omega)}\right)^2|\zeta-\zeta_0|^2}+\right.$$

$$\left.+\frac{|\tilde{r}'(e^{i\varphi/2}+\omega e^{i\varphi/2})|\,\rho}{\left(\frac{\omega(2+\omega)}{2(1+\omega)}\right)^2|\zeta-\zeta_0|^2}+\frac{|\tilde{r}''(e^{i\varphi/2}+\omega e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho^2}{2\rho^2|\zeta-\zeta_0|^2}\right)\leqslant$$

$$\leqslant\frac{\mathrm{const}}{|\zeta-\zeta_0|^2},\quad \zeta\in Z_0^+\bigcap D_{\varepsilon_0}.$$

Thus

$$\iint\limits_{Z_0^+\bigcap(D_{\varepsilon_0}\setminus D_\varepsilon)} \left|\frac{r(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant$$

$$\leqslant \mathrm{const}\int\limits_{\varepsilon}^{\varepsilon_0}\frac{d\rho}{\rho} = \mathrm{const}\ln\frac{\varepsilon_0}{\varepsilon}.$$

Setting $\varepsilon = \frac{1}{|t|}$ yields $I_3 = O(\ln(|t|))$, as $t\to\infty$.

(B) Now let $\dfrac{1}{1+\omega} \leqslant 2\varepsilon_0$.
If $\zeta \in Z_0^+\bigcap D_{\varepsilon_0}$, then the following estimates hold

$$|\zeta\pm\zeta_1| \geqslant \frac{1}{2\varepsilon_0} - 1 - \varepsilon_0, \quad |\zeta\pm\zeta_2| \geqslant \frac{1}{2\varepsilon_0} - 3\varepsilon_0, \quad |\zeta+\zeta_0| > 2-\varepsilon_0.$$

Consequently,

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}}{|\zeta|^4}|\zeta-\zeta_0|, \quad \zeta\in Z_0^+\bigcap D_{\varepsilon_0}.$$

and the part of the integral I_{ext} over Z_0^+ for this case can be estimated, proceeding as in the previous section, as $O\left(\frac{\ln(|t|)}{|t|}\right)$, $t\to\infty$.
If $\zeta \in Z_2^+\bigcap D_{\varepsilon_0}$, then the following estimates hold

$$|\zeta\pm\zeta_1| \geqslant 1-3\varepsilon_0, \quad |\zeta\pm\zeta_0| \geqslant 1-3\varepsilon_0,$$

and thus

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}}{|\zeta|^4}|\zeta-\zeta_2||\zeta+\zeta_2|.$$

We can estimate $|\zeta+\zeta_2| \geqslant |\zeta_2| = \dfrac{1}{1+\omega}$. Now let us expand $r(\zeta)$ into Taylor formula in the neighborhood of $\frac{1}{1+\omega}e^{i\varphi/2}$:

$$r(\zeta) = r\left(\frac{1}{1+\omega}e^{i\varphi/2}\right) + r'\left(\frac{1}{1+\omega}e^{i\varphi/2}+\lambda\rho e^{i\theta}\right)\rho, \quad \lambda\in[0,1].$$

For an arbitrary value of $\omega>0$ (satisfying $\frac{1}{1+\omega}\leqslant 2\varepsilon_0$) the following estimate holds:

$$\left|r\left(\frac{1}{1+\omega}e^{i\varphi/2}\right)\right| \leqslant \mathrm{const}\left|\frac{1}{1+\omega}\right|^N.$$

This yields

$$\left|\frac{r(\zeta)}{S'_\zeta(u,\zeta)}\right| \leqslant \text{const}\left(\frac{|r(\frac{1}{1+\omega}e^{i\varphi/2})|}{\frac{1}{1+\omega}|\zeta-\zeta_2|}+\frac{|r'(\frac{1}{1+\omega}e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho}{\rho|\zeta-\zeta_2|}\right)\leqslant\frac{\text{const}}{|\zeta-\zeta_2|}.$$

In the same manner

$$\left|\frac{r(\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{|r(\zeta)||\zeta|^8}{|\zeta+\zeta_2|^2|\zeta-\zeta_2|^2} \leqslant \text{const}\left(\frac{|r(\frac{1}{1+\omega}e^{i\varphi/2})|}{\left(\frac{1}{1+\omega}\right)^2|\zeta-\zeta_2|^2}+\right.$$

$$\left.+\frac{|r'(\frac{1}{1+\omega}e^{i\varphi/2})|\,\rho}{\left(\frac{1}{1+\omega}\right)^2|\zeta-\zeta_2|^2}+\frac{|r''(\frac{1}{1+\omega}e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho^2}{2\rho^2|\zeta-\zeta_2|^2}\right)\leqslant\frac{\text{const}}{|\zeta-\zeta_2|^2}.$$

Following further the reasoning from the case (A) we obtain that

$$I_{ext} = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \text{as} \quad t\to\infty$$

uniformly on $u\in\mathbb{C}$. □

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